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A nondegeneracy result for least energy solutions to a biharmonic problem with nearly critical exponent

Dedicated to professor Toshitaka Nagai on the occasion of his sixties birthday

By

Tomohiko SATO * and Futoshi TAKAHASHI **

Abstract

Consider the problem $\Delta^2 u = c_0 K(x) u^{p_\varepsilon}$, $u > 0$ in Ω , $u = \Delta u = 0$ on $\partial\Omega$, where Ω is a smooth bounded domain in \mathbb{R}^N ($N \geq 5$), $c_0 = (N-4)(N-2)N(N+2)$, $p = (N+4)/(N-4)$, $p_\varepsilon = p - \varepsilon$ and K is a smooth positive function on $\overline{\Omega}$.

Under some assumptions on the coefficient function K , which include the nondegeneracy of its unique maximum point as a critical point of $\text{Hess}K$, we prove that least energy solutions of the problem are nondegenerate for $\varepsilon > 0$ small.

§ 1. Introduction

Consider the problem

$$(1.1) \quad \begin{cases} \Delta^2 u = c_0 K(x) u^{p_\varepsilon} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ ($N \geq 5$) is a smooth bounded domain, $c_0 = (N-4)(N-2)N(N+2)$, $p_\varepsilon = p - \varepsilon$, $p = (N+4)/(N-4)$ is the critical Sobolev exponent with respect to the Sobolev embedding $H^2 \cap H_0^1(\Omega) \hookrightarrow L^{p+1}(\Omega)$, and $\varepsilon > 0$ is a small parameter. Here, K is a positive function in $C^2(\overline{\Omega})$.

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We put an assumption on the coefficient function K :

(K): $K \in C^2(\overline{\Omega})$, $0 < K(x) \leq 1$, $K^{-1}(\max_{\overline{\Omega}} K) = \{x_0\} \subset \Omega$ with $K(x_0) = 1$, and x_0 is a nondegenerate critical point of K .

In the following, as solutions of (1.1) we consider only least energy solutions u_ε such that

$$\frac{\int_{\Omega} |\Delta u_\varepsilon|^2 dx}{\left(\int_{\Omega} K(x) |u_\varepsilon|^{p_\varepsilon+1} dx\right)^{\frac{2}{p_\varepsilon+1}}} = \inf_{u \in H^2 \cap H_0^1(\Omega)} \frac{\int_{\Omega} |\Delta u|^2 dx}{\left(\int_{\Omega} K(x) |u|^{p_\varepsilon+1} dx\right)^{\frac{2}{p_\varepsilon+1}}}.$$

We easily check that least energy solutions blow up in the sense that $\|u_\varepsilon\|_{L^\infty(\Omega)} = u_\varepsilon(x_\varepsilon) \rightarrow +\infty$ as $\varepsilon \rightarrow 0$, and that the maximum point x_ε of u_ε converges to a maximum point of K in $\overline{\Omega}$. Therefore we have $x_\varepsilon \rightarrow x_0$ as $\varepsilon \rightarrow 0$, here by the assumption (K), x_0 is the unique interior maximum point of K .

In this note, we prove the nondegeneracy of least energy solutions to (1.1) when $\varepsilon > 0$ is sufficiently small, under the assumption (K). Here as usual, the nondegeneracy of u_ε for small ε means that the problem

$$(1.2) \quad \begin{cases} \Delta^2 v_\varepsilon = c_0 p_\varepsilon K(x) u_\varepsilon^{p_\varepsilon-1} v_\varepsilon & \text{in } \Omega, \\ v_\varepsilon = \Delta v_\varepsilon = 0 & \text{on } \partial\Omega \end{cases}$$

admits no solution except for the trivial one for $\varepsilon > 0$ small enough.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^N$ ($N \geq 5$) be a smooth bounded domain. Under the assumption (K), least energy solution u_ε to (1.1) is nondegenerate for $\varepsilon > 0$ small.*

The precise asymptotic behavior of least energy solutions as $\varepsilon \rightarrow 0$ when $K \not\equiv 1$ was obtained in [6] under the assumption (K). Using this result, we prove Theorem 1.1 along the line of [7] and [8], the original idea of which comes from [4].

§ 2. Preliminaries

In this section, we recall some facts which are needed in the sequel. Let $G = G(x, z)$ denote the Green function of Δ^2 under the Navier boundary condition:

$$\begin{cases} \Delta^2 G(\cdot, z) = \delta_z & \text{in } \Omega, \\ G(\cdot, z) = \Delta G(\cdot, z) = 0 & \text{on } \partial\Omega. \end{cases}$$

We decompose G as $G(x, z) = \Gamma(x, z) - H(x, z)$, where $\Gamma(x, z)$ is the fundamental solution of Δ^2 :

$$\Gamma(x, z) = \begin{cases} \frac{1}{(N-4)(N-2)\sigma_N} |x - z|^{4-N}, & N \geq 5, \\ \frac{1}{\sigma_4} \log |x - z|^{-1}, & N = 4, \end{cases}$$

where σ_N is the volume of the $(N - 1)$ dimensional unit sphere in \mathbb{R}^N and $H(x, z)$ is the regular part of the Green function. Finally, let $R(z) = H(z, z)$ denote the Robin function of the Green function of Δ^2 with the Navier boundary condition. By the maximum principle, we have $R > 0$ on Ω and $R(z) \rightarrow +\infty$ as z tends to the boundary of Ω . In the following, we set $\overline{G} = -\Delta G$. Then \overline{G} is the Green function of $-\Delta$ under the Dirichlet boundary condition, and satisfy

$$\begin{cases} -\Delta G = \overline{G}, & -\Delta \overline{G} = \delta_z & \text{in } \Omega, \\ G > 0, \overline{G} > 0 & & \text{in } \Omega, \\ G = \overline{G} = 0 & & \text{on } \partial\Omega. \end{cases}$$

Lemma 2.1. *For any $z \in \Omega$, there holds*

$$(2.1) \quad \int_{\partial\Omega} ((x - z) \cdot \nu_x) \left(\frac{\partial G}{\partial \nu_x} \right) \left(\frac{\partial \overline{G}}{\partial \nu_x} \right) (x, z) ds_x = (N - 4)R(z),$$

$$(2.2) \quad \int_{\partial\Omega} \frac{\partial G}{\partial \nu_x}(x, z) \frac{\partial \overline{G}}{\partial \nu_x}(x, z) \nu_i(x) ds_x = \frac{\partial R}{\partial z_i}(z), \quad (i = 1, \dots, N),$$

$$\int_{\partial\Omega} \left(\frac{\partial \overline{G}}{\partial x_i} \right) \frac{\partial}{\partial z_j} \left(\frac{\partial G}{\partial \nu_x} \right) (x, z) ds_x + \int_{\partial\Omega} \left(\frac{\partial G}{\partial x_i} \right) \frac{\partial}{\partial z_j} \left(\frac{\partial \overline{G}}{\partial \nu_x} \right) (x, z) ds_x$$

$$(2.3) \quad = \frac{\partial^2 R}{\partial z_i \partial z_j}(z), \quad (i, j = 1, \dots, N).$$

Here ν_x is the outer unit normal at $x \in \partial\Omega$.

Proof. See [3]:Lemma 3.1 and Lemma 3.3. Note that our sign convention is different from that of [3]. By differentiating (2.2) with respect to z_j , noting that $\left(\frac{\partial G}{\partial \nu_x}(x, z) \right) \nu_i(x) = \frac{\partial G}{\partial x_i}(x, z)$, $\left(\frac{\partial \overline{G}}{\partial \nu_x}(x, z) \right) \nu_i(x) = \frac{\partial \overline{G}}{\partial x_i}(x, z)$ on $\partial\Omega$, we see that (2.3) holds. \square

Lemma 2.2. *Let u_ε be a solution to (1.1) and v_ε be a solution to (1.2). Denote $\overline{u}_\varepsilon = -\Delta u_\varepsilon$ and $\overline{v}_\varepsilon = -\Delta v_\varepsilon$. Then the following identities hold true:*

$$(2.4) \quad \int_{\partial\Omega} ((x - z) \cdot \nu_x) \left\{ \left(\frac{\partial u_\varepsilon}{\partial \nu_x} \right) \left(\frac{\partial \overline{v}_\varepsilon}{\partial \nu_x} \right) + \left(\frac{\partial \overline{u}_\varepsilon}{\partial \nu_x} \right) \left(\frac{\partial v_\varepsilon}{\partial \nu_x} \right) \right\} ds_x$$

$$= c_0 \int_{\Omega} u_\varepsilon^{p_\varepsilon} v_\varepsilon (x - z) \cdot \nabla K(x) dx$$

for any $z \in \mathbb{R}^N$ and

$$(2.5) \quad \int_{\partial\Omega} \left\{ \left(\frac{\partial \overline{u}_\varepsilon}{\partial x_i} \right) \left(\frac{\partial v_\varepsilon}{\partial \nu_x} \right) + \left(\frac{\partial u_\varepsilon}{\partial x_i} \right) \left(\frac{\partial \overline{v}_\varepsilon}{\partial \nu_x} \right) \right\} ds_x = c_0 \int_{\Omega} \left(\frac{\partial K}{\partial x_i} \right) u_\varepsilon^{p_\varepsilon} v_\varepsilon dx$$

for $i = 1, 2, \dots, N$.

Proof. For smooth f, g , we have the formula

$$(2.6) \quad \begin{aligned} & \int_{\Omega} ((\Delta^2 f)g - (\Delta^2 g)f) dx \\ &= \int_{\partial\Omega} \left(\frac{\partial \Delta f}{\partial \nu_x} \right) g - \left(\frac{\partial \Delta g}{\partial \nu_x} \right) f ds_x + \int_{\partial\Omega} \left(\frac{\partial f}{\partial \nu_x} \right) \Delta g - \left(\frac{\partial g}{\partial \nu_x} \right) \Delta f ds_x. \end{aligned}$$

Set $w_\varepsilon(x) = (x - z) \cdot \nabla u_\varepsilon(x) + \alpha_\varepsilon u_\varepsilon(x)$ where $\alpha_\varepsilon = \frac{4}{p_\varepsilon - 1}$. Direct computation yields that

$$\Delta^2 w_\varepsilon = (\alpha_\varepsilon + 4)c_0 K(x) u_\varepsilon^{p_\varepsilon} + c_0 p_\varepsilon K(x) u_\varepsilon^{p_\varepsilon - 1} (x - z) \cdot \nabla u_\varepsilon + c_0 u_\varepsilon^{p_\varepsilon} (x - z) \cdot \nabla K(x).$$

Since v_ε satisfies $\Delta^2 v_\varepsilon = c_0 p_\varepsilon K(x) u_\varepsilon^{p_\varepsilon - 1} v_\varepsilon$, we have

$$(\Delta^2 w_\varepsilon) v_\varepsilon - (\Delta^2 v_\varepsilon) w_\varepsilon = (\alpha_\varepsilon + 4 - p_\varepsilon \alpha_\varepsilon) c_0 K(x) u_\varepsilon^{p_\varepsilon} v_\varepsilon + c_0 u_\varepsilon^{p_\varepsilon} (x - z) \cdot \nabla K(x) v_\varepsilon = 0.$$

Integrating this identity on Ω with the formula (2.6), and noting that

$$w_\varepsilon(x) = (x - z) \cdot \nu_x \left(\frac{\partial u_\varepsilon}{\partial \nu_x} \right), \quad \Delta w_\varepsilon(x) = (x - z) \cdot \nu_x \left(\frac{\partial \Delta u_\varepsilon}{\partial \nu_x} \right)$$

for $x \in \partial\Omega$, we have (2.4).

On the other hand, differentiating the equation in (1.1) with respect to x_i , we have

$$\Delta^2 \left(\frac{\partial u_\varepsilon}{\partial x_i} \right) = c_0 p_\varepsilon K(x) u_\varepsilon^{p_\varepsilon - 1} \left(\frac{\partial u_\varepsilon}{\partial x_i} \right) + c_0 \left(\frac{\partial K}{\partial x_i} \right) u_\varepsilon^{p_\varepsilon} \quad \text{in } \Omega.$$

Multiplying this by v_ε , and the equation of v_ε by $\left(\frac{\partial u_\varepsilon}{\partial x_i} \right)$ and subtracting, we obtain

$$(\Delta^2 v_\varepsilon) \left(\frac{\partial u_\varepsilon}{\partial x_i} \right) - \left(\Delta^2 \left(\frac{\partial u_\varepsilon}{\partial x_i} \right) \right) v_\varepsilon = 0.$$

Finally, integration by parts formula (2.6) yields (2.5). \square

Next is the asymptotic result by [6]. In what follows, we use a symbol $\|\cdot\|$ to denote the L^∞ norm of functions.

Theorem 2.3. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 5$ be a smooth bounded domain. Let u_ε be a least energy solution to (1.1) for $\varepsilon > 0$ and let $x_\varepsilon \in \Omega$ be a point such that $u_\varepsilon(x_\varepsilon) = \|u_\varepsilon\|$. Assume (K). Then after passing to a subsequence, the following estimate holds true:*

There exists a constant $C > 0$ independent of ε such that for any $R_\varepsilon \rightarrow \infty$ with $r_\varepsilon = R_\varepsilon \|u_\varepsilon\|^{-\frac{p_\varepsilon - 1}{4}} \rightarrow 0$,

$$(2.7) \quad \begin{cases} u_\varepsilon(x) \leq C \frac{\|u_\varepsilon\|}{\left(1 + \|u_\varepsilon\|^{\frac{4}{N-4}} |x - x_\varepsilon|^2\right)^{\frac{N-4}{2}}}, & \text{for } |x - x_\varepsilon| \leq r_\varepsilon, \\ u_\varepsilon(x) \leq \frac{C}{\|u_\varepsilon\|} \frac{1}{|x - x_\varepsilon|^{N-4}}, & \text{for } \{|x - x_\varepsilon| > r_\varepsilon\} \cap \Omega. \end{cases}$$

Furthermore, as $\varepsilon \rightarrow 0$,

$$(2.8) \quad (1) \begin{cases} |x_\varepsilon - x_0| = O(\|u_\varepsilon\|^{-2}) & N = 5, \\ |x_\varepsilon - x_0| = o(\|u_\varepsilon\|^{-\frac{2}{N-4}}) & N \geq 6, \end{cases}$$

$$(2.9) \quad (2) \|u_\varepsilon\|^\varepsilon \rightarrow 1,$$

$$(2.10) \quad (3) \|u_\varepsilon\| u_\varepsilon(x) \rightarrow 2(N-4)(N-2)\sigma_N G(x, x_0) \quad \text{in } C_{loc}^3(\overline{\Omega} \setminus \{x_0\}),$$

$$(2.11) \quad (4) \begin{cases} \varepsilon \|u_\varepsilon\|^2 \rightarrow \frac{2^{15}}{21} \pi R(x_0) & N = 5, \\ \varepsilon \|u_\varepsilon\|^2 \rightarrow -\frac{1}{4} \Delta K(x_0) + 480\pi^3 R(x_0) & N = 6, \\ \varepsilon \|u_\varepsilon\|^{\frac{4}{N-4}} \rightarrow -\frac{2}{(N-2)(N-4)} \Delta K(x_0) & N \geq 7. \end{cases}$$

Now, consider the scaled function

$$(2.12) \quad \tilde{u}_\varepsilon(y) := \frac{1}{\|u_\varepsilon\|} u_\varepsilon \left(\frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon-1}{4}}} + x_\varepsilon \right), \quad y \in \Omega_\varepsilon := \|u_\varepsilon\|^{\frac{p_\varepsilon-1}{4}} (\Omega - \varepsilon).$$

\tilde{u}_ε satisfies $0 < \tilde{u}_\varepsilon \leq 1$, $\tilde{u}_\varepsilon(0) = 1$, and

$$\begin{cases} \Delta^2 \tilde{u}_\varepsilon = c_0 K\left(\frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon-1}{4}}} + x_\varepsilon\right) \tilde{u}_\varepsilon^{p_\varepsilon} & \text{in } \Omega_\varepsilon, \\ \tilde{u}_\varepsilon = \Delta \tilde{u}_\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon. \end{cases}$$

Since $\|u_\varepsilon\| \rightarrow \infty$ as $\varepsilon \rightarrow 0$ and x_ε does not approach to $\partial\Omega$, we see $\Omega_\varepsilon \rightarrow \mathbb{R}^N$. By standard elliptic estimates, we have a subsequence denoted also by \tilde{u}_ε that

$$(2.13) \quad \tilde{u}_\varepsilon \rightarrow U \quad \text{compact uniformly in } \mathbb{R}^N$$

as $\varepsilon \rightarrow 0$ for some function U . Passing to the limit, we obtain that U is a solution of

$$\begin{cases} \Delta^2 U = c_0 U^p & \text{in } \mathbb{R}^N, \\ 0 < U \leq 1, \quad U(0) = 1, \\ \lim_{|y| \rightarrow \infty} U(y) = 0. \end{cases}$$

According to the uniqueness theorem by Chang Shou Lin [5], we obtain

$$(2.14) \quad U(y) = \left(\frac{1}{1 + |y|^2} \right)^{\frac{N-4}{2}}.$$

In terms of \tilde{u}_ε in (2.12), the estimate (2.7) reads

$$(2.15) \quad \tilde{u}_\varepsilon(y) \leq \begin{cases} CU(y) & \text{for } |y| \leq R_\varepsilon, \\ C \frac{1}{|y|^{N-4}} & \text{for } \{|y| > R_\varepsilon\} \cap \Omega_\varepsilon, \end{cases}$$

where $R_\varepsilon \rightarrow \infty$ is any sequence as in the above.

Here, we recall a theorem by Bartsch, Weth and Willem [1].

Lemma 2.4. *Let v_0 be a solution to*

$$\begin{cases} \Delta^2 v_0 = c_0 p U^{p-1} v_0 & \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |\Delta v_0|^2 dy < \infty. \end{cases}$$

Then there exist a_j ($j = 1, 2, \dots, N$), $b \in \mathbb{R}$ such that v_0 can be written as

$$v_0 = \sum_{j=1}^N a_j \frac{y_j}{(1 + |y|^2)^{(N-2)/2}} + b \frac{1 - |y|^2}{(1 + |y|^2)^{(N-2)/2}}.$$

§ 3. Proof of Theorem 1.1

In this section, we will prove Theorem 1.1.

We argue by contradiction. We assume there exists a non-trivial solution v_ε to (1.2) satisfying $\|v_\varepsilon\| = \|u_\varepsilon\|$ for any $\varepsilon > 0$.

Consider the scaled function

$$(3.1) \quad \tilde{v}_\varepsilon(y) = \frac{1}{\|u_\varepsilon\|} v_\varepsilon \left(\frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon-1}{4}}} + x_\varepsilon \right), \quad y \in \Omega_\varepsilon = \|u_\varepsilon\|^{\frac{p_\varepsilon-1}{4}} (\Omega - x_\varepsilon).$$

We see $0 < \tilde{v}_\varepsilon \leq 1$ and \tilde{v}_ε satisfies

$$(3.2) \quad \begin{cases} \Delta^2 \tilde{v}_\varepsilon = c_0 p_\varepsilon K \left(\frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon-1}{4}}} + x_\varepsilon \right) \tilde{u}_\varepsilon^{p_\varepsilon-1} \tilde{v}_\varepsilon & \text{in } \Omega_\varepsilon, \\ \tilde{v}_\varepsilon = \Delta \tilde{v}_\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon, \\ \|\tilde{v}_\varepsilon\|_{L^\infty(\Omega_\varepsilon)} = 1. \end{cases}$$

By $\|\tilde{v}_\varepsilon\|_{L^\infty(\Omega_\varepsilon)} = 1$, elliptic estimate implies that

$$(3.3) \quad \tilde{v}_\varepsilon \rightarrow v_0 \quad \text{compact uniformly in } \mathbb{R}^N$$

for some v_0 and v_0 satisfies

$$\Delta^2 v_0 = c_0 p U^{p-1} v_0 \quad \text{in } \mathbb{R}^N.$$

Also by arguing as in [7], we have

$$(3.4) \quad \int_{\Omega_\varepsilon} |\Delta \tilde{v}_\varepsilon|^2 dy \leq C$$

for some $C > 0$ independent of $\varepsilon > 0$ small. By (3.4) and Fatou's lemma, we also have

$$\int_{\mathbb{R}^N} |\Delta v_0|^2 dy \leq C.$$

Thus by Lemma 2.4, we have

$$(3.5) \quad v_0 = \sum_{j=1}^N a_j \frac{y_j}{(1 + |y|^2)^{(N-2)/2}} + b \frac{1 - |y|^2}{(1 + |y|^2)^{(N-2)/2}}.$$

In the following, we divide the proof into three steps.

Step 1. $b = 0$.

Step 2. $a_j = 0, j = 1, \dots, N$.

Step 3. $v_0 = 0$ leads to a contradiction.

First, by using the Kelvin transformation and a local supremum estimate for weak solutions to a linear biharmonic equation by Caristi and Mitidieri [2], we can obtain the pointwise estimate for the scaled function \tilde{v}_ε , just as in [7] Lemma 3.1.

Lemma 3.1. *Let \tilde{v}_ε be a solution of (3.2). Then we have the estimate*

$$(3.6) \quad |\tilde{v}_\varepsilon(y)| \leq CU(y), \quad \forall y \in \Omega_\varepsilon$$

for some $C > 0$.

Also by Lemma 3.1 and Theorem 2.3 (2.7), we have the following convergence result. For a proof, see Lemma 3.2 in [7].

Lemma 3.2. *Let $\omega \subset \Omega$ be any neighborhood of $\partial\Omega$ not containing x_0 . Then we have*

$$(3.7) \quad \|u_\varepsilon\|_{v_\varepsilon} \rightarrow -2(N-2)(N-4)\sigma_N bG(\cdot, x_0) \quad \text{in } C^3(\omega).$$

Proof of Step 1. Here, we prove only the case $N \geq 7$. Proof of the cases $N = 5$ and $N = 6$ will be done by a similar argument; see [8] for the second order $-\Delta$ case.

Putting $z = x_0$ in (2.4) and multiplying $\|u_\varepsilon\|^{4/(N-4)}$, we have

$$(3.8) \quad \begin{aligned} & \|u_\varepsilon\|^{\frac{4}{N-4}-2} \int_{\partial\Omega} ((x - x_0) \cdot \nu_x) \left(\frac{\partial \|u_\varepsilon\| u_\varepsilon}{\partial \nu_x} \right) \left(\frac{\partial \|u_\varepsilon\| \bar{v}_\varepsilon}{\partial \nu_x} \right) ds_x \\ & + \|u_\varepsilon\|^{\frac{4}{N-4}-2} \int_{\partial\Omega} ((x - x_0) \cdot \nu_x) \left(\frac{\partial \|u_\varepsilon\| \bar{u}_\varepsilon}{\partial \nu_x} \right) \left(\frac{\partial \|u_\varepsilon\| v_\varepsilon}{\partial \nu_x} \right) ds_x \\ & = \|u_\varepsilon\|^{\frac{4}{N-4}} c_0 \int_{\Omega} u_\varepsilon^{p_\varepsilon} v_\varepsilon (x - x_0) \cdot \nabla K(x) dx. \end{aligned}$$

As $\frac{4}{N-4} < 2$ if $N \geq 7$, LHS of (3.8) converges to 0 as $\varepsilon \rightarrow 0$. On the other hand, by Taylor's formula and the change of variables, we write

$$(\text{RHS}) \text{ of (3.8)} =: C_1 + C_2 + C_3 + C_4$$

where, putting $b_{ij} = \frac{\partial^2 K}{\partial x_i \partial x_j}(x_0)$,

$$\begin{aligned} C_1 &= c_0 \|u_\varepsilon\|^{\frac{4}{N-4}+p_\varepsilon+1-(\frac{p_\varepsilon-1}{4})N-(\frac{p_\varepsilon-1}{2})} \int_{\Omega_\varepsilon} \tilde{u}_\varepsilon^{p_\varepsilon} \tilde{v}_\varepsilon(y) \sum_{i,j=1}^N b_{ij} y_i y_j dy, \\ C_2 &= 2c_0 \|u_\varepsilon\|^{\frac{4}{N-4}+p_\varepsilon+1-(\frac{p_\varepsilon-1}{4})N-(\frac{p_\varepsilon-1}{4})} \int_{\Omega_\varepsilon} \tilde{u}_\varepsilon^{p_\varepsilon} \tilde{v}_\varepsilon(y) \sum_{i,j=1}^N b_{ij} y_i (x_{\varepsilon j} - x_{0j}) dy, \\ C_3 &= c_0 \|u_\varepsilon\|^{\frac{4}{N-4}+p_\varepsilon+1-(\frac{p_\varepsilon-1}{4})N} \int_{\Omega_\varepsilon} \tilde{u}_\varepsilon^{p_\varepsilon} \tilde{v}_\varepsilon(y) \sum_{i,j=1}^N b_{ij} (x_{\varepsilon i} - x_{0i})(x_{\varepsilon j} - x_{0j}) dy, \\ C_4 &= c_0 \|u_\varepsilon\|^{\frac{4}{N-4}+p_\varepsilon+1-(\frac{p_\varepsilon-1}{4})N} \int_{\Omega_\varepsilon} \tilde{u}_\varepsilon^{p_\varepsilon} \tilde{v}_\varepsilon(y) \left(O \left| \frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon-1}{4}}} + x_\varepsilon - x_0 \right|^3 \right) dy. \end{aligned}$$

By (2.15), (3.6), (2.9), (2.8) and the dominated convergence theorem, we see

$$\begin{aligned} C_2 &= O(\|u_\varepsilon\|^{\frac{2}{N-4}+\frac{N-3}{4}\varepsilon}) \times O\left(\int_{\mathbb{R}^N} U^p v_0(y) |y| dy + o(1)\right) \times o(\|u_\varepsilon\|^{-\frac{2}{N-4}}) = o(1), \\ C_3 &= O(\|u_\varepsilon\|^{\frac{4}{N-4}+\frac{N-4}{4}\varepsilon}) \times O\left(\int_{\mathbb{R}^N} U^p v_0(y) dy + o(1)\right) \times o(\|u_\varepsilon\|^{-\frac{4}{N-4}}) = o(1), \\ C_4 &= O(\|u_\varepsilon\|^{\frac{4}{N-4}+\frac{N-4}{4}\varepsilon}) \int_{\Omega_\varepsilon} \tilde{u}_\varepsilon^{p_\varepsilon} \tilde{v}_\varepsilon(y) \left(O \left(\left| \frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon-1}{2}}} \right|^3 \right) + O(|x_\varepsilon - x_0|^3) \right) \\ &= O(\|u_\varepsilon\|^{\frac{4}{N-4}}) \times O(\|u_\varepsilon\|^{-\frac{6}{N-4}}) \times O\left(\int_{\mathbb{R}^N} U^p v_0(y) (|y|^3 + 1) dy + o(1)\right) \\ &= O(\|u_\varepsilon\|^{-\frac{2}{N-4}}) \end{aligned}$$

as $\varepsilon \rightarrow 0$. As for C_1 , we see

$$\begin{aligned} C_1 &= c_0 \|u_\varepsilon\|^{(\frac{N-2}{4})\varepsilon} \int_{\Omega_\varepsilon} \tilde{u}_\varepsilon^{p_\varepsilon} \tilde{v}_\varepsilon(y) \sum_{i,j=1}^N b_{ij} y_i y_j dy \\ &\rightarrow c_0 \int_{\mathbb{R}^N} U^p(y) v_0(y) \sum_{i,j=1}^N b_{ij} y_i y_j dy = \frac{c_0}{N} b \Delta K(x_0) \int_{\mathbb{R}^N} U^p(y) \frac{1 - |y|^2}{(1 + |y|^2)^{(N-2)/2}} |y|^2 dy. \end{aligned}$$

Thus letting $\varepsilon \rightarrow 0$ in (3.8), we have

$$0 = \Delta K(x_0) \times b.$$

Hence we obtain $b = 0$, because our nondegeneracy assumption of x_0 assures that $\Delta K(x_0) < 0$ strictly. \square

Proof of Step 2.

In this step, we prove $a_j = 0, j = 1, 2, \dots, N$ in (3.5) by using the next lemma.

Lemma 3.3. *Assume $b = 0$ and $a = (a_1, \dots, a_N) \neq 0$ in (3.5). Then we have*

$$\|u_\varepsilon\|^{\frac{N-2}{N-4}} v_\varepsilon \rightarrow 2(N-2)\sigma_N \sum_{j=1}^N a_j \left(\frac{\partial G}{\partial z_j}(x, z) \right) \Big|_{z=x_0}$$

in $C_{loc}^3(\overline{\Omega} \setminus \{x_0\})$.

Proof. Since $-\Delta \bar{v}_\varepsilon = c_0 p_\varepsilon K(x) u_\varepsilon^{p_\varepsilon-1} v_\varepsilon$ in Ω , $\bar{v}_\varepsilon = 0$ on $\partial\Omega$, the Green representation formula implies that

$$(3.9) \quad \bar{v}_\varepsilon(x) = c_0 p_\varepsilon \int_{\Omega} \bar{G}(x, z) K(z) u_\varepsilon^{p_\varepsilon-1}(z) v_\varepsilon(z) dz$$

for any $x \in \overline{\Omega} \setminus \{x_0\}$, here $\bar{G}(x, z) = -\Delta_x G(x, z)$ is the Green function of $-\Delta$ under the Dirichlet boundary condition. By a change of variables, we see

$$\begin{aligned} c_0 p_\varepsilon \int_{\Omega} \bar{G}(x, z) K(z) u_\varepsilon^{p_\varepsilon-1}(z) v_\varepsilon(z) dz \\ = c_0 p_\varepsilon \|u_\varepsilon\|^{p_\varepsilon - (\frac{p_\varepsilon-1}{4})N} \int_{\Omega_\varepsilon} \bar{G}_\varepsilon(x, y) K_\varepsilon(y) \tilde{u}_\varepsilon^{p_\varepsilon-1} \tilde{v}_\varepsilon(y) dy \end{aligned}$$

where $\bar{G}_\varepsilon(x, y) = \bar{G}(x, \frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon-1}{4}}} + x_\varepsilon)$ and $K_\varepsilon(y) = K(\frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon-1}{4}}} + x_\varepsilon)$. By (2.13) and (3.3), we obtain

$$K_\varepsilon(y) \tilde{u}_\varepsilon^{p_\varepsilon-1} \tilde{v}_\varepsilon(y) \rightarrow \sum_{j=1}^N a_j \left(\frac{\partial}{\partial y_j} \frac{-1}{(N+4)} U^p(y) \right)$$

uniformly on compact subsets of \mathbb{R}^N .

Now, let us consider the following linear first order PDE

$$\sum_{j=1}^N a_j \frac{\partial w_\varepsilon}{\partial y_j} = \tilde{u}_\varepsilon^{p_\varepsilon-1} \tilde{v}_\varepsilon(y), \quad y \in \mathbb{R}^N$$

with the initial condition $w_\varepsilon|_{\Gamma_a} = \frac{-1}{(N+4)} U^p(y)$, where $\Gamma_a = \{x \in \mathbb{R}^N | x \cdot a = 0\}$. Here, the right hand side is assumed to be 0 outside of Ω_ε . By the unique solvability, we have the solution w_ε of this problem with the estimate $w_\varepsilon(y) = O(|y|^{-(N+3)})$ as $|y| \rightarrow \infty$, since $\tilde{u}_\varepsilon^{p_\varepsilon-1} \tilde{v}_\varepsilon(y) = O(|y|^{-(N+4)})$ by (2.15) and (3.6). Also we have

$$w_\varepsilon \rightarrow \frac{-1}{(N+4)} U^p \quad \text{uniformly on compact subsets on } \mathbb{R}^N$$

and

$$\int_{\Omega_\varepsilon} w_\varepsilon(y) dy \rightarrow \frac{-1}{(N+4)} \int_{\mathbb{R}^N} U^p dy = \left(\frac{-1}{N+4} \right) \left(\frac{2\sigma_N}{N(N+2)} \right)$$

by the dominated convergence theorem. Using integration by parts, we have

$$\begin{aligned}
\bar{v}_\varepsilon(x) &= c_0 p_\varepsilon \|u_\varepsilon\|^{p_\varepsilon - (\frac{p_\varepsilon-1}{4})N} \int_{\Omega_\varepsilon} \bar{G}_\varepsilon(x, y) K_\varepsilon(y) \sum_{j=1}^N a_j \frac{\partial w_\varepsilon}{\partial y_j} dy \\
&= -c_0 p_\varepsilon \|u_\varepsilon\|^{p_\varepsilon - (\frac{p_\varepsilon-1}{4})N} \sum_{j=1}^N a_j \int_{\Omega_\varepsilon} \frac{\partial}{\partial y_j} \{ \bar{G}_\varepsilon(x, y) K_\varepsilon(y) \} w_\varepsilon(y) dy \\
&= -c_0 p_\varepsilon \|u_\varepsilon\|^{p_\varepsilon - (\frac{p_\varepsilon-1}{4})N - (\frac{p_\varepsilon-1}{4})} \sum_{j=1}^N a_j \int_{\Omega_\varepsilon} \frac{\partial}{\partial z_j} \{ \bar{G}(x, z) K(z) \} \Big|_{z = \frac{y}{\frac{p_\varepsilon-1}{4}} + x_\varepsilon} w_\varepsilon(y) dy.
\end{aligned}$$

Note that $p_\varepsilon - (\frac{p_\varepsilon-1}{4})N - (\frac{p_\varepsilon-1}{4}) = -(\frac{N-2}{N-4}) + \varepsilon(\frac{N-3}{4})$. Now, we see

$$\begin{aligned}
&\frac{\partial}{\partial z_j} \{ \bar{G}(x, z) K(z) \} \Big|_{z = \frac{y}{\frac{p_\varepsilon-1}{4}} + x_\varepsilon} \\
&\rightarrow \left(\frac{\partial \bar{G}}{\partial z_j}(x, x_0) \right) K(x_0) + \bar{G}(x, x_0) \left(\frac{\partial K}{\partial z_j}(x_0) \right) \\
&= \frac{\partial \bar{G}}{\partial z_j}(x, x_0)
\end{aligned}$$

uniformly on compact subsets of \mathbb{R}^N as $\varepsilon \rightarrow 0$, since x_0 is a critical point of K with $K(x_0) = 1$. Therefore, we have the convergence

$$\begin{aligned}
\|u_\varepsilon\|^{\frac{N-2}{N-4}} \bar{v}_\varepsilon(x) &\rightarrow -c_0 p \left(\frac{-1}{N+4} \right) \left(\frac{2\sigma_N}{N(N+2)} \right) \sum_{j=1}^N a_j \left(\frac{\partial \bar{G}}{\partial z_j}(x, z) \right) \Big|_{z=x_0} \\
&= 2(N-2)\sigma_N \sum_{j=1}^N a_j \left(\frac{\partial \bar{G}}{\partial z_j}(x, z) \right) \Big|_{z=x_0}
\end{aligned}$$

for any $x \in \bar{\Omega} \setminus \{x_0\}$. Elliptic estimates implies this convergence holds true in $C_{loc}^1(\bar{\Omega} \setminus \{x_0\})$. This proves Lemma. \square

Now, assume the contrary that $a = (a_1, \dots, a_N) \neq 0$. We multiply both sides of (2.5) in Lemma 2.2 by $\|u_\varepsilon\|^{(N-2)/(N-4)} \times \|u_\varepsilon\|^{-1}$ to get

$$\begin{aligned}
&\|u_\varepsilon\|^{-2} \left[\int_{\partial\Omega} \left(\frac{\partial \|u_\varepsilon\| \bar{u}_\varepsilon}{\partial x_i} \right) \left(\frac{\partial \|u_\varepsilon\|^{\frac{N-2}{N-4}} v_\varepsilon}{\partial \nu_x} \right) ds_x + \left(\frac{\partial \|u_\varepsilon\| u_\varepsilon}{\partial x_i} \right) \left(\frac{\partial \|u_\varepsilon\|^{\frac{N-2}{N-4}} \bar{v}_\varepsilon}{\partial \nu_x} \right) ds_x \right] \\
(3.10) \quad &= \|u_\varepsilon\|^{-1 + \frac{N-2}{N-4}} c_0 \int_{\Omega} \left(\frac{\partial K}{\partial x_i} \right) u_\varepsilon^{p_\varepsilon} v_\varepsilon dx
\end{aligned}$$

As $\varepsilon \rightarrow 0$, we see that

$$\int_{\partial\Omega} \left(\frac{\partial \|u_\varepsilon\| \bar{u}_\varepsilon}{\partial x_i} \right) \left(\frac{\partial \|u_\varepsilon\|^{\frac{N-2}{N-4}} v_\varepsilon}{\partial \nu_x} \right) ds_x + \left(\frac{\partial \|u_\varepsilon\| u_\varepsilon}{\partial x_i} \right) \left(\frac{\partial \|u_\varepsilon\|^{\frac{N-2}{N-4}} \bar{v}_\varepsilon}{\partial \nu_x} \right) ds_x$$

tends to

$$\begin{aligned} & 4(N-4)(N-2)^2 \sigma_N^2 \sum_{j=1}^N a_j \times \\ & \int_{\partial\Omega} \left\{ \left(\frac{\partial \bar{G}}{\partial x_i} \right) \frac{\partial}{\partial \nu_x} \left(\frac{\partial G}{\partial z_j} \right) (x, x_0) + \left(\frac{\partial G}{\partial x_i} \right) \frac{\partial}{\partial \nu_x} \left(\frac{\partial \bar{G}}{\partial z_j} \right) (x, x_0) \right\} ds_x \\ & = 4(N-4)(N-2)^2 \sigma_N^2 \sum_{j=1}^N a_j \frac{\partial^2 R}{\partial z_i \partial z_j} (z) \Big|_{z=x_0}, \end{aligned}$$

here we have used Theorem 2.3 (2.10), Lemma 3.3 and Lemma 2.1 (2.3). Thus we have (LHS) of (3.10) tends to 0 as $\varepsilon \rightarrow 0$.

On the other hand, again we solve the linear PDE

$$(3.11) \quad \sum_{j=1}^N a_j \frac{\partial w_\varepsilon}{\partial y_j} = \tilde{u}_\varepsilon^{p_\varepsilon} \tilde{v}_\varepsilon(y), \quad y \in \mathbb{R}^N$$

with the initial condition $w_\varepsilon|_{\Gamma_a} = \frac{-1}{2N} U^{p+1}(y)$, where $\Gamma_a = \{x \in \mathbb{R}^N | x \cdot a = 0\}$. Here as before, the RHS of (3.11) is understood as 0 outside of Ω_ε . The solution w_ε satisfies the estimate $w_\varepsilon(y) = O(|y|^{-2N+1})$ as $|y| \rightarrow \infty$, since $\tilde{u}_\varepsilon^{p_\varepsilon} \tilde{v}_\varepsilon(y) = O(U^{p_\varepsilon+1}(y)) = O(|y|^{-2N})$ by (2.15) and (3.6). As before, we have

$$w_\varepsilon \rightarrow \frac{-1}{2N} U^{p+1} \quad \text{uniformly on compact subsets on } \mathbb{R}^N$$

and

$$\int_{\Omega_\varepsilon} w_\varepsilon(y) dy \rightarrow \frac{-1}{2N} \int_{\mathbb{R}^N} U^{p+1} dy = \frac{-1}{2N} \sigma_N C_N$$

by the dominated convergence theorem, where $C_N = \int_0^\infty \frac{r^{N-1}}{(1+r^2)^N} dr = \frac{\Gamma(N/2)^2}{2\Gamma(N)}$. Thus,

(RHS of (2.5)) $\times \|u_\varepsilon\|^{\frac{N-2}{N-4}-1}$ is

$$\begin{aligned}
& c_0 \|u_\varepsilon\|^{-1+\frac{N-2}{N-4}} \int_{\Omega} \left(\frac{\partial K}{\partial x_i} \right) u_\varepsilon^{p_\varepsilon} v_\varepsilon dx \\
&= c_0 \|u_\varepsilon\|^{\frac{N-2}{N-4}+p_\varepsilon-(\frac{p_\varepsilon-1}{4})N} \int_{\Omega_\varepsilon} \left(\frac{\partial K}{\partial x_i} \right) \left(\frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon-1}{4}}} + x_\varepsilon \right) \tilde{u}_\varepsilon^{p_\varepsilon} \tilde{v}_\varepsilon dy \\
&= c_0 \|u_\varepsilon\|^{\frac{N-2}{N-4}+p_\varepsilon-(\frac{p_\varepsilon-1}{4})N} \int_{\Omega_\varepsilon} \left(\frac{\partial K}{\partial x_i} \right) \left(\frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon-1}{4}}} + x_\varepsilon \right) \sum_{j=1}^N a_j \frac{\partial w_\varepsilon}{\partial y_j} dy \\
&= -c_0 \|u_\varepsilon\|^{\frac{N-2}{N-4}+p_\varepsilon-(\frac{p_\varepsilon-1}{4})N} \sum_{j=1}^N a_j \int_{\Omega_\varepsilon} \frac{\partial}{\partial y_j} \left\{ \left(\frac{\partial K}{\partial x_i} \right) \left(\frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon-1}{4}}} + x_\varepsilon \right) \right\} w_\varepsilon(y) dy \\
&= -c_0 \|u_\varepsilon\|^{\frac{N-2}{N-4}+p_\varepsilon-(\frac{p_\varepsilon-1}{4})N-(\frac{p_\varepsilon-1}{4})} \sum_{j=1}^N a_j \int_{\Omega_\varepsilon} \left(\frac{\partial^2 K}{\partial x_i \partial x_j} \right) (x) \Big|_{x=\frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon-1}{4}}}+x_\varepsilon} w_\varepsilon(y) dy \\
&\rightarrow -c_0 \left(\lim_{\varepsilon \rightarrow 0} \|u_\varepsilon\|^{(\frac{N-3}{4})\varepsilon} \right) \sum_{j=1}^N a_j \frac{\partial^2 K}{\partial x_i \partial x_j} (x_0) \left(\lim_{\varepsilon \rightarrow 0} \int_{\Omega_\varepsilon} w_\varepsilon(y) dy \right) \\
&= \frac{1}{2N} c_0 \sigma_N C_N \sum_{j=1}^N a_j \frac{\partial^2 K}{\partial x_i \partial x_j} (x_0).
\end{aligned}$$

Thus we have

$$\sum_{j=1}^N a_j \frac{\partial^2 K}{\partial x_i \partial x_j} (x_0) = 0.$$

By our assumption of the nondegeneracy of x_0 , the matrix $\left(\frac{\partial^2 K}{\partial x_i \partial x_j} \right) (x_0)$ is invertible. Therefore we obtain that $a_j = 0$ for all $j = 1, \dots, N$. Thus we have proved Step 2. \square

Proof of Step 3.

By Step 1 and Step 2, we have obtained that the limit function $\lim_{\varepsilon \rightarrow 0} \tilde{v}_\varepsilon = v_0 \equiv 0$. Since $\|\tilde{v}_\varepsilon\|_{L^\infty(\Omega_\varepsilon)} = 1$, there exists $y_\varepsilon \in \Omega_\varepsilon$ such that $\tilde{v}_\varepsilon(y_\varepsilon) = 1$ and $|y_\varepsilon| \rightarrow \infty$, because the above convergence $\tilde{v}_\varepsilon \rightarrow v_0 \equiv 0$ is uniform on compact sets of \mathbb{R}^N . But this is not possible because of Lemma 3.1. This proves Theorem 1.1. \square

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